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Linear Stability of Self Similar Flow: 7. The Primakoff Blast Wave and its Generalizations

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The linear stability of those Sedov blast wave similarity solutions for which the flow is homologous behind the shock, of which the best-known example is the Primakoff point blast model, along with that of their two-dimensional counterparts, is proved analytically. This conclusion in the three-dimensional case applies to all perturbations, and in the two-dimensional case to flutelike ($k_z = 0$) modes. Implications for supernova remnant evolution are discussed.

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CONTENTS

1. Introduction

In the course of supernova remnant (SNR) evolution, there appear to be four well-defined stages: the first $\sim 10^2$ years of unimpeded expansion of hot gas away from the collapsed central object; a period $\sim 2\times10^4$ years during which this envelope sweeps up interstellar material, slowing down somewhat as it does so; a period $\sim 5\times10^5$ years during which line emission is the most effective cooling mechanism; and a terminal stage in which the expanding material becomes indistinguishable from the background gas.

During the second stage, a shock wave is launched which propagates ahead of the expanding shell. In the present paper, we are concerned with the hydrodynamic character of this shock wave. Other workers (see, for example, Chevalier 1976) have carried out elaborate one-dimensional calculations incorporating realistic transport, radiation and atomic physics models. While these are indispensable for acquiring a complete understanding of SNR behavior, idealized simple models are also very useful. They afford accurate predictions over a restricted parameter range and qualitative descriptions over a wider range.

Such a model has been derived by Sedov (1946; 1959) for a blast wave expanding from a point source into a gas-filled surrounding region. [An abbreviated description has been given by Landau and Lifshitz (1959); see also Newman (1977). Taylor (1950) and Von Neumann (1947) independently developed similar models in studies of atmospheric explosions.] The model consists of a similarity solution of the ideal gas equations for a strong shock (Mach number $M = V/c_s >> 1$, where V is the shock speed and c_s the velocity of sound in the undisturbed medium) produced by deposition of

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an explosion energy W at the origin in a medium of density $\bar{\rho}$. This model can be expected to be fairly accurate in the second stage of SNR evolution, during which the shock is effectively strong and line radiation processes negligible. Chevalier (1976) and others do in fact utilize it in this way in studying the evolution of Type II supernovas.

A question which comes readily to mind is whether the Sedov solutions are stable, particularly when $\bar{\rho}$ is allowed to vary with position. In their general form the solutions are quite messy. The fluid variables ρ, v and p are given implicitly as functions of r and t through the similarity variable $Wt^2/\bar{\rho}r^5$ (for uniform $\bar{\rho}$). Hence investigation of the linear stability of perturbations about these solutions is likely to be impossible in the general case except through numerical approximations.

It is not a trivial exercise to demonstrate stability even of plane shocks, a problem which has been treated by D'yakov (1954) and Erpenbeck (1962) in ideal gas-dynamic systems, and by Gardner and Kruskal (1964) for MHD shocks. All of these calculations found that shock waves propagating in a uniform medium are stable. The physical mechanism is easy to describe. A small ripple in the shock front gives rise to divergence (convergence) in the curved portion ahead (behind) the main front. These regions become weaker (stronger) than the unperturbed shock, hence propagate slower (faster) than average, thus reducing the amplitude of the ripple. Evidently the longer the perturbation wavelength, the weaker the stabilizing effect of this mechanism.

When this reasoning is applied to spherical shock waves it becomes obscure and must be regarded as no better than a plausibility argument. If the density $\bar{\rho}$ drops sufficiently rapidly with increasing radius, it is possible that the ripples ahead of the shock run away, while those behind fall farther behind, leading to instability. Lerche and Vasyliunas (1976) and Isenberg (1977) have claimed that for at least some decreasing power-law density distributions, Sedov shocks are unstable for any value of the adiabatic index γ. Their conclusions, obtained after very elaborate analysis, were remarkable in predicting instability at short wavelengths, contradicting the intuitive argument appropriate to the planar case. Newman (1979) has disputed the results, on the ground that Lerche and Vasyliunas (1976) and Isenberg (1977) improperly treated the boundary condition at the shock front.

The present paper is addressed to the problem of determining the stability of a restricted class of the Sedov solutions, those in which the outward flow behind the shock in homologous. This type of solution was apparently first considered by Primakoff (see Courant and Friedrichs 1948), who was studying underwater explosions. He found that for $\bar{\rho}$ = const, the ideal fluid equations have an explicit similarity solution when $\gamma = 7$, the approximate value of the effective adiabatic index of water at high pressure. Keller (1956) generalized the results to arbitrary power-law undisturbed density profiles and determined the value of the density exponent q corresponding to each choice of $\gamma > 1$.

Using a formalism originally developed by Bernstein and Book (1978) and Book and Bernstein (1979) to study the Rayleigh-Taylor instability of homologous expansions and contractions, we solve the linearized equations of motion exactly. Our criterion of stability is that the amplitude of the shock front perturbations divided by the shock radius vanish as $t \to \infty$.

We are able to show that the Primakoff blast wave and its three-dimensional generalizations are stable against all modes, while the corresponding two-dimensional line blast wave solutions are stable against all flute-like (independent of z) perturbations (the case $k_z \neq 0$ is not amenable to treatment).

The plan of the paper is as follows. In Section 2 we review the derivation of the generalized Primakoff model, using an approach similar to that of Keller (1956). In Section 3 we derive the linearized equations describing the evolution of a small perturbation about the basic state. Invoking the boundary conditions obtained from the Rankine-Hugoniot relations across the shock reduces the calculations to solution of an eigenvalue problem. The results are presented in Section 4 for cylindrical and spherical blast waves. In Section 5 we discuss briefly the implications of our results.

2. Generalized Primakoff blast wave model

The density ρ, velocity $\mathbf{v}$, and pressure p for an ideal polytrope with adiabatic index γ satisfy the equations

$$\dot{\rho} + \rho \nabla \cdot \mathbf{v} = 0; \tag{2.1}$$

$$\rho \dot{\mathbf{v}} + \nabla p = 0; \tag{2.2}$$

$$\dot{p} + \gamma p \nabla \cdot \mathbf{v} = 0, \tag{2.3}$$

where a dot denotes the material derivative $\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla$. On a surface $\mathcal{S}$ with normal $\mathbf{n}$, moving with velocity $\mathbf{V}$ (where both $\mathbf{n}$ and $\mathbf{V}$ can depend on position), the fluid variables can change discontinuously according to the Rankine-Hugoniot ("jump") conditions

$$\langle \rho \mathbf{n} \cdot (\mathbf{V} - \mathbf{v}) \rangle = 0; \tag{2.4}$$

$$\langle \rho \mathbf{n} \cdot (\mathbf{V} - \mathbf{v}) (\mathbf{V} - \mathbf{v}) + p \mathbf{n} \rangle = 0; \tag{2.5}$$

$$\langle [\rho (V - v)^2 + \frac{2\gamma p}{\gamma - 1}] \mathbf{n} \cdot (\mathbf{V} - \mathbf{v}) \rangle = 0, \tag{2.6}$$

where < > denotes the jump in the quantity enclosed. By dotting and crossing (2.5) with $\mathbf{n}$, we find the projections

$$\langle \rho [\mathbf{n} \cdot (\mathbf{V} - \mathbf{v})]^2 + p \rangle = 0 \tag{2.7}$$

and

$$\langle \rho \mathbf{n} \cdot (\mathbf{V} - \mathbf{v}) [\mathbf{n} \times (\mathbf{V} - \mathbf{v})] \rangle = 0. \tag{2.8}$$

The latter, by virtue of (2.4), reduces to

$$\langle \mathbf{n} \times (\mathbf{V} - \mathbf{v}) \rangle = 0. \tag{2.9}$$

We will use bars to distinguish quantities ahead of $\mathcal{S}$ from those behind. Let us assume that $\bar{\mathbf{v}} = 0$, i.e., the fluid in front of $\mathcal{S}$ is stationary. It follows that $\mathbf{n} \times \mathbf{v} = 0$, so the velocity behind $\mathcal{S}$ lies in the direction of $\mathbf{n}$. Hence only the magnitudes

$v = \underset{\sim}{n} \cdot \underset{\sim}{v}$ and $V = \underset{\sim}{n} \cdot \underset{\sim}{V}$ enter into the jump conditions, and we can rewrite (2.4) - (2.6) as

$$\rho(V - v) = \bar{\rho}\, V; \tag{2.10}$$

$$\rho\, (V - v)^2 + p = \bar{\rho}\, V^2 + \bar{p}\ ; \tag{2.11}$$

$$\rho\, (V - v)^3 + \frac{2\gamma p}{\gamma-1}\, (V - v) = \bar{\rho}\, V^3 + \frac{2\gamma \bar{p}}{\gamma-1}\, V. \tag{2.12}$$

In the limit of very strong shocks, effectively $\bar{p} \to 0$ and the solutions of (2.10) - (2.12) simplify to

$$\frac{\rho}{\bar{\rho}} = \frac{\gamma+1}{\gamma-1}; \tag{2.13}$$

$$\frac{v}{V} = \frac{2}{\gamma+1}\ ; \tag{2.14}$$

$$p = \frac{2}{\gamma+1}\, \bar{\rho}\, V^2 = \bar{\rho}\, v\, V. \tag{2.15}$$

Behind the shock front $\mathcal{S}$ we seek solutions of (2.1) - (2.3) consistent with homologous (uniform) expansion,

$$R = r\, f(t). \tag{2.16}$$

Here R is the position of a particular element of fluid at t, and r is the position that element would occupy at some time t_0 (when f = 1); by assumption the function f is the same for all fluid elements. There is no special physical significance to the choice of the Lagrangian variable r. We could, for example, label each element instead by the position it occupied prior to being overtaken by the shock, but use of r simplifies the analysis.

For symmetric one-dimensional motion, Eqs. (2.1) - (2.3) become

$$\dot{\rho} + \rho R^{1-\nu}\, \frac{\partial}{\partial R}(R^{\nu-1} v) = 0; \tag{2.17}$$

$$\rho\,\dot{v} + \frac{\partial p}{\partial R} = 0; \tag{2.18}$$

$$\dot{p} + \gamma\, p\, R^{1-\nu} \frac{\partial}{\partial R}\left(R^{\nu-1} v\right) = 0, \tag{2.19}$$

where $\nu = 1, 2, 3$ for planar, cylindrical and spherical geometry, respectively. Using the Lagrangian description in terms of r and t, we find from (2.17)

$$\rho(r,t) = \rho_0(r) f^{-\nu}, \tag{2.20}$$

while from (2.19),

$$p(r,t) = p_0(r)\; f^{-\nu\gamma}. \tag{2.21}$$

Substitution into (2.18) now yields an equation which separates into a spatial and a temporal o.d.e., viz.,

$$\frac{dp_0}{dr} = \omega^2 r \rho_0 \tag{2.22}$$

and

$$\ddot{f}\; f^{\;\nu(\gamma-1)+1} = -\,\omega^2. \tag{2.23}$$

Equation (2.23) can be integrated twice, yielding

$$\dot{f}^{\,2} = \frac{2\omega^2}{\nu(\gamma-1)} f^{\,\nu(1-\gamma)} \tag{2.24}$$

and

$$f = \left\{1 + \left[\frac{\nu(\gamma-1)}{2} + 1\right]\left[\frac{2}{\nu(\gamma-1)}\right]^{\frac{1}{2}} \omega(t-t_o)\right\}^{\frac{2}{\nu(\gamma-1)+2}}. \tag{2.25}$$

From (2.16) the velocity satisfies

$$v = \dot{R} = r\dot{f} = R\dot{f}/f. \tag{2.26}$$

We denote the radius of the shock at time t by S(t). When the jump condition (2.14) is applied at R = S, we obtain

$$\frac{S\dot{f}}{f} = \frac{2\dot{S}}{\gamma+1}, \tag{2.27}$$

which integrates to

$$S = s\, f^{\frac{\gamma+1}{2}} , \tag{2.28}$$

where s is the shock position at $t = t_0$. Hence from (2.13),

$$\rho(S,t) = \rho_0(S/f)f^{-\nu} = \frac{\gamma+1}{\gamma-1}\,\bar{\rho} . \tag{2.29}$$

Suppose $\bar{\rho}$ has a power-law dependence on R. In that case we can write

$$\bar{\rho} = \bar{\rho}_0 \;(R/s\;)^q \;\equiv\; QR^q, \tag{2.30}$$

$\bar{\rho}_0$, Q and q constant. From (2.28) - (2.30),

$$\rho_0(\frac{S}{f}) = \frac{\gamma+1}{\gamma-1}\,\bar{\rho}_0 f^{\,\nu}(\frac{S}{s})^q = \frac{\gamma+1}{\gamma-1}\,\bar{\rho}_0 f^{\,\nu+q}\;(\frac{S}{s\;f})^q$$

$$= \frac{\gamma+1}{\gamma-1}\,\bar{\rho}_0\;(\frac{S}{s\;f})^{\frac{2(\nu+q)}{\gamma-1}+q} , \tag{2.31}$$

whence for arbitrary r

$$\rho_0(r) = \frac{\gamma+1}{\gamma-1}\,\bar{\rho}_0\;(\frac{r}{s})^{\frac{2(\nu+q)}{\gamma-1}+q} . \tag{2.32}$$

Similarly, from (2.15),

$$p(S,t) = p_0(S/f)f^{-\nu\gamma} = \frac{2\bar{\rho}}{\gamma+1}\,\dot{S}^2$$

$$= \frac{2\bar{\rho}_0}{\gamma+1}\;(\frac{S}{s})^q\;s^2\;f^{\gamma-1}\;\dot{f}^2\;(\frac{\gamma+1}{2})^2. \tag{2.33}$$

So

$$p_0\left(\frac{S}{f}\right) = \frac{2\bar{\rho}_0}{\gamma+1}\,\frac{2\omega^2}{\nu(\gamma-1)}\left(\frac{\gamma+1}{2}\right)^2 s^2 \left(\frac{S}{s}\right)^{q} f^{\gamma-1+\nu\gamma+\nu(1-\gamma)}$$

$$= \frac{(\gamma+1)\,\bar{\rho}_0\, s^2\,\omega^2}{\nu(\gamma-1)} \left(\frac{S}{s}\right)_f^{q} f^{\gamma+\nu-1+q}$$

$$= \frac{(\gamma+1)\,\bar{\rho}_0\, s^2\,\omega^2}{\nu(\gamma-1)} \left(\frac{S}{s}\right)_f^{q+\frac{2(\gamma+\nu-1+q)}{\gamma-1}}, \tag{2.34}$$

whence

$$p_0(r) = \frac{(\gamma+1)\,\bar{\rho}_0\, s^2\,\omega^2}{\nu(\gamma-1)} \left(\frac{r}{s}\right)^{q+\frac{2(\gamma+\nu-1+q)}{\gamma-1}}. \tag{2.35}$$

Substitution of (2.32) and (2.35) in (2.22) results in the requirement that

$$q = \frac{\gamma(\nu-2) - 3\nu+2}{\gamma+1}. \tag{2.36}$$

It turns out to be convenient to write all equations in terms of γ, using (2.36) to eliminate q. Thus (2.32) and (2.35) reduce to

$$\rho_0(r) = \frac{\gamma+1}{\gamma-1}\,\bar{\rho}_0\left(\frac{r}{s}\right)^{\nu-2} \tag{2.37}$$

and

$$p_0(r) = \frac{\gamma+1}{\nu(\gamma-1)}\,\bar{\rho}_0\, s^2\,\omega^2\left(\frac{r}{s}\right)^{\nu}, \tag{2.38}$$

respectively. Table 1 lists the values of q for two- and three-dimensional flows corresponding to some typical choices of γ. Note that for $\nu=3$ the medium ahead of the shock is uniform (q=0) when $\gamma=7$, the original Primakoff solution. For $\nu=2$, however, the undisturbed medium is always nonuniform except in the incompressible limit $\gamma \to \infty$.

Table 1

γ	q	
	$\nu = 2$	$\nu = 3$
1	-2	-3
5/3	-3/2	-2
3	-1	-1
7	-1/2	0
∞	0	1

The derivation of the equations describing a self-similar blast wave was first carried out in terms of Lagrangian variables by Keller (1956), who obtained a slightly more general result. The present approach has the virtue of yielding an explicit solution for the fluid variables as functions of r and t, amounting to a special case of the better known Sedov (1946) solution, which is cast in Eulerian variables and is in general implicit. We can recover the Eulerian form of the present results as follows. If we choose t_0 so that the zero of time coincides with the instant of explosion, (2.25) becomes

$$f = \left(\left\{ \left[\frac{2}{\nu(\gamma-1)}\right]^{\frac{1}{2}} + \left[\frac{\nu(\gamma-1)}{2}\right]^{\frac{1}{2}} \right\} \omega t \right)^{\frac{2}{\nu(\gamma-1)+2}}$$

$$= C\,(\omega t)^{\frac{2}{\nu(\gamma-1)+2}} , \tag{2.39}$$

where

$$C^{\nu(\gamma-1)+2} = \frac{[\nu(\gamma-1)+2]^2}{2\,\nu(\gamma-1)} . \tag{2.40}$$

Then from (2.20) and (2.37), the Eulerian form of ρ is

$$\rho(R,t) = \frac{\gamma+1}{\gamma-1}\,\rho_0\, C^{2-2\nu} \left(\frac{R}{s}\right)^{\nu-2} (\omega t)^{\frac{4(1-\nu)}{\nu(\gamma-1)+2}} . \tag{2.41}$$

From (2.26),

$$v(R,t) = \frac{2}{\nu(\gamma-1)+2}\,\frac{R}{t} . \tag{2.42}$$

From (2.21) and (2.38),

$$p(R,t) = \frac{\gamma+1}{\nu(\gamma-1)}\,\bar{\rho}_0\,\omega^2 s^2 \left(\frac{R}{s}\right)^2 C^{-\nu(\gamma+1)} (\omega t)^{-\frac{2\nu(\gamma+1)}{\nu(\gamma-1)+2}}$$

$$= \frac{2(\gamma-1)\rho}{[\nu(\gamma-1)+2]^2} \frac{R^2}{t^2} \tag{2.43}$$

by (2.40) and (2.41).

Equations (2.41) - (2.43) constitute an Eulerian description of the flow behind the shock. Note that s and ω occur only in the expression $s^{2-\nu} \omega^{\frac{4(1-\nu)}{\nu(\gamma-1)+2}}$.

It is useful to rewrite (2.41) - (2.43) in terms of Q and the total energy W,

$$W = \pi 2^{\nu-1} \int_0^S dR\, R^{\nu-1} \left(\frac{1}{2} \rho v^2 + \frac{p}{\gamma-1}\right) , \tag{2.44}$$

instead of the less easily interpreted s and ω. For this purpose we evaluate (2.44) [note that, by (2.43), the two terms in the integrand are equal], with the result

$$W = \frac{2^{\nu-1}\pi}{\nu^2} \frac{\gamma+1}{(\gamma-1)^2} Q\, \omega^2\, s^{\frac{2[\nu(\gamma-1)+2]}{\gamma+1}} . \tag{2.45}$$

Then we can rewrite (2.41) as

$$\rho(R,t) = \frac{\gamma+1}{\gamma-1} \left\{ \frac{\nu(\gamma-1)[\nu(\gamma-1)+2]^2}{2^\nu \pi\ (\gamma+1)} \right\}^{\frac{2(1-\nu)}{\nu(\gamma-1)+2}}$$

$$\cdot\ Q^{\frac{\nu(\gamma+1)}{\nu(\gamma-1)+2}}\ W^{\frac{2(1-\nu)}{\nu(\gamma-1)+2}}\ R^{\nu-2}\ t^{\frac{4(1-\nu)}{\nu(\gamma-1)+2}} , \tag{2.46}$$

with a similar expression for p by way of (2.43). This form of the solution is identical with that given by Sedov (1946; 1959).

3. Linearized equations of motion

We follow Bernstein and Book (1978) in obtaining linearized equations for the development of a small perturbation about the solutions of Section 2. The element of fluid whose trajectory is given by $\underset{\sim}{R}(\underset{\sim}{r},t)$ is assumed to undergo a displacement to $\underset{\sim}{R}(\underset{\sim}{r},t) + \underset{\sim}{\xi}(r,t)$. The perturbation $\underset{\sim}{\xi}$ satisfies the linearized form of (2.2),

$$\rho\ddot{\underset{\sim}{\xi}} + \rho_1\ddot{\underset{\sim}{R}} + \nabla_{\underset{\sim}{R}}p_1 - (\nabla_{\underset{\sim}{R}}\underset{\sim}{\xi}) \cdot \nabla_{\underset{\sim}{R}}p = 0, \tag{3.1}$$

where first-order quantities are distinguished by the subscript 1. Substituting the perturbed velocity, density and pressure from

$$\underset{\sim}{v}_1 = \dot{\underset{\sim}{\xi}}\ , \tag{3.2}$$

$$\rho_1 = -\ \rho\nabla_{\underset{\sim}{R}} \cdot \underset{\sim}{\xi} \tag{3.3}$$

and

$$p_1 = -\ \gamma p\nabla_{\underset{\sim}{R}} \cdot \underset{\sim}{\xi}, \tag{3.4}$$

and making use of the relation (2.16), we obtain

$$\omega^{-2}f^{\nu(\gamma-1)+2}\ \ddot{\underset{\sim}{\xi}} + \underset{\sim}{r}\nabla\cdot\dot{\underset{\sim}{\xi}} - \frac{\gamma}{\nu}r^{2-\nu}\ \nabla(r^{\nu}\nabla\cdot\underset{\sim}{\xi}) - (\nabla\underset{\sim}{\xi})\cdot\underset{\sim}{r} = 0, \tag{3.5}$$

where ∇ denotes the gradient with respect to $\underset{\sim}{r}$.

We seek solutions of (3.5) by the method of separation of variables. Substituting

$$\underset{\sim}{\xi}(\underset{\sim}{r},t) = \underset{\sim}{X}(\underset{\sim}{r})T(t), \tag{3.6}$$

we find

$$f^{\nu(\gamma-1)+2}\,\ddot{T} = -\,\mu\omega^2 T \tag{3.7}$$

and

$$\mu\underset{\sim}{X} - \underset{\sim}{r}\sigma + \frac{\gamma}{\nu}\left[\nabla(r^2\sigma) + (\nu-2)\underset{\sim}{r}\sigma\right] + \nabla(\underset{\sim}{r}\cdot\underset{\sim}{X}) - \underset{\sim}{X} = 0, \tag{3.8}$$

where μ is a separation constant and $\sigma = \nabla\cdot\underset{\sim}{X}$. Substituting for f from (2.25) converts (3.7) into

$$\left\{\left[\frac{2}{\nu(\gamma-1)}\right]^{\frac{1}{2}} + \left[\frac{\nu(\gamma-1)}{2}\right]^{\frac{1}{2}}\right\}^2 t^2\ddot{T} = -\,\mu T, \tag{3.9}$$

which is homogeneous in t. Equation (3.8) is likewise homogeneous in r for $\nu = 3$, and also for $\nu = 2$ if $\underset{\sim}{X}$ is independent of z, the coordinate in the direction of the axis of symmetry. It follows that r and t both enter into $\underset{\sim}{\xi}$ with power-law dependences, viz.,

$$\underset{\sim}{X} \sim r^{\alpha}, \tag{3.10}$$

$$T \sim t^{\beta}. \tag{3.11}$$

By (2.16), the solutions expressed in terms of the Eulerian variables $\underset{\sim}{R}$,t have the same property (with different coefficients α', β').

Writing

$$X_r = ar^{\alpha}, \tag{3.12}$$

$$r\sigma = br^{\alpha}, \tag{3.13}$$

where X_r is the radial component of $\underset{\sim}{X}$, we see that (3.8) yields

$$(\mu+\alpha)\;a + \left[\frac{\gamma}{\nu}(\alpha+\nu-1)-1\right]\,b = 0. \tag{3.14}$$

A second equation connecting a and b results when we take the divergence of (3.8):

$$[(\alpha+1)(\alpha+2) - \Lambda]a + \Big\{\mu-1 + [\frac{(\nu-2)\gamma}{\nu} -1](\nu+\alpha-1)$$

$$+ \frac{\gamma}{\nu}[(\alpha+1)(\alpha+2) - \Lambda]\Big\} b = 0. \tag{3.15}$$

Here Λ is the coefficient of the terms in the Laplacian resulting from the angular dependence of $\underset{\sim}{\zeta}$. For $\nu = 3$, $\Lambda = \ell(\ell+1)$, while for $\nu = 2$, $\Lambda = m^2$, where ℓ and m have their usual meanings.

From (3.9) and (3.11) we get a relation between μ and β,

$$\mu = -\frac{[\nu(\gamma-1)+2]^2 \beta(\beta-1)}{2\nu(\gamma-1)} . \tag{3.16}$$

In order to proceed further, we need to apply appropriate boundary conditions to the solution.

As a result of the perturbation, the shock front $\mathcal{S}$ undergoes a small displacement into $\mathcal{S}'$. We can describe this by saying that at time t a point on $\mathcal{S}$ at $\underset{\sim}{S}(t)$ is shifted to

$$\underset{\sim}{S}'(t) = \underset{\sim}{S} + \underset{\sim}{\zeta}(\underset{\sim}{S},t). \tag{3.17}$$

There is an arbitrariness in the definition of $\underset{\sim}{\zeta}$, only the normal component of which is significant. We thus free to define $\underset{\sim}{\zeta}$ in the most convenient manner, as a mapping along the unperturbed normal $\underset{\sim}{n}$:

$$\underset{\sim}{\zeta}(\underset{\sim}{S},t) = \underset{\sim}{n}(\underset{\sim}{S},t)\ \zeta(\underset{\sim}{S},t) \tag{3.18}$$

We introduce $\underset{\sim}{n}_1$ and $\underset{\sim}{V}_1$, the first order corrections to $\underset{\sim}{n}$ and $\underset{\sim}{V}$, respectively, and label the perturbed fluid variables with primes to indicate that they are evaluated on $\mathcal{S}'$ instead of $\mathcal{S}$. In first order, the jump conditions (2.13) - (2.15) yield

$$\rho_1' = 0; \tag{3.19}$$

$$\underset{\sim}{n} \cdot \underset{\sim}{v}_1' + \underset{\sim}{n}_1 \cdot \underset{\sim}{v} = \frac{2}{\gamma+1}(\underset{\sim}{n} \cdot \underset{\sim}{V}_1 + \underset{\sim}{n}_1 \cdot \underset{\sim}{V}); \tag{3.20}$$

$$p_1' = \frac{4\bar{\rho}}{\gamma+1}(\underset{\sim}{n} \cdot \underset{\sim}{V})(\underset{\sim}{n} \cdot \underset{\sim}{V}_1 + \underset{\sim}{n}_1 \cdot \underset{\sim}{V}). \tag{3.21}$$

From kinematic arguments, $\underset{\sim}{n}_1$ is readily shown to have the form

$$\underset{\sim}{n}_1 = \underset{\sim}{n}\,\underset{\sim}{n} \cdot (\nabla \underset{\sim}{\zeta}) \cdot \underset{\sim}{n} - (\nabla \underset{\sim}{\zeta}) \cdot \underset{\sim}{n}, \tag{3.22}$$

from which it follows that $\underset{\sim}{n} \cdot \underset{\sim}{n}_1 = 0$. Thus $\underset{\sim}{n}_1$ is orthogonal to $\underset{\sim}{n}$, and therefore to $\underset{\sim}{v}$ and $\underset{\sim}{V}$. Equations (3.20) and (3.21) simplify to

$$\underset{\sim}{n} \cdot \underset{\sim}{v}_1' = \frac{2}{\gamma+1}\, \underset{\sim}{n} \cdot \underset{\sim}{V}_1 \tag{3.23}$$

and

$$p_1' = \frac{4\bar{\rho}(\underset{\sim}{n} \cdot \underset{\sim}{V})(\underset{\sim}{n} \cdot \underset{\sim}{V}_1)}{\gamma+1} = 2\bar{\rho}\,(\underset{\sim}{n} \cdot \underset{\sim}{V})(\underset{\sim}{n} \cdot \underset{\sim}{v}_1'). \tag{3.24}$$

The expressions (3.2) - (3.4) for the perturbed fluid variables must now be replaced by their Eulerian counterparts,

$$\rho_1 = -\rho\nabla_{\underset{\sim}{R}} \cdot \underset{\sim}{\xi} - \underset{\sim}{\xi} \cdot \nabla_{\underset{\sim}{R}}\rho; \tag{3.25}$$

$$\underset{\sim}{v}_1 = \dot{\underset{\sim}{\xi}} - \underset{\sim}{\xi} \cdot \nabla_{\underset{\sim}{R}}\underset{\sim}{v}; \tag{3.26}$$

$$p_1 = -\gamma p\nabla_{\underset{\sim}{R}} \cdot \underset{\sim}{\xi} - \underset{\sim}{\xi} \cdot \nabla_{\underset{\sim}{R}}p. \tag{3.27}$$

Here $\underset{\sim}{\xi} = \underset{\sim}{\xi}(\underset{\sim}{R},t)$ is the displacement in position experienced by a fluid element whose unperturbed position at time t was $\underset{\sim}{R}$. To evaluate the perturbed quantities at $\underset{\sim}{S}'$, we expand in Taylor series to obtain

$$\rho'_1(\underline{S},t) = \rho_1(\underline{S},t) + \underline{\zeta}\cdot\nabla_{\underline{R}}\,\rho(\underline{S},t); \tag{3.28}$$

$$\underline{V}'_1(\underline{S},t) = \underline{V}_1(\underline{S},t) + \underline{\zeta}\cdot\nabla_{\underline{R}}\underline{V}; \tag{3.29}$$

$$p'_1(\underline{S},t) = p_1(\underline{S},t) + \underline{\zeta}\cdot\nabla p. \tag{3.30}$$

Combining (3.25) - (3.30) and substituting in (3.19) and (3.23) - (3.24), we find three equations containing $\underline{\zeta}$, $\underline{V}_1$ and $\underline{\xi}$. For a spherical unperturbed shock front $\mathcal{S}$ these take the form

$$(\zeta-\xi_r)\frac{\partial\rho}{\partial R} - \rho\Sigma = \frac{\gamma+1}{\gamma-1}\,\zeta\,\frac{\partial\bar{\rho}}{\partial R}; \tag{3.31}$$

$$\dot{\xi}_r + (\zeta-\xi_r)\frac{\partial v}{\partial R} = \frac{2V_{1r}}{\gamma+1}\ ; \tag{3.32}$$

$$-\gamma p\Sigma + (\zeta-\xi_r)\frac{\partial p}{\partial R} = \frac{4\bar{\rho}VV_{1r}}{\gamma+1} + \frac{2V^2\zeta}{\gamma+1}\,\frac{\partial\bar{\rho}}{\partial R}, \tag{3.33}$$

all evaluated at $R = S$. Here ξ_r and V_{1r} are the radial components of $\underline{\xi}$ and $\underline{V}_1$, and $\Sigma = \nabla_{\underline{R}}\cdot\underline{\xi}$. Instead of solving for $\underline{\xi}$ in general, we use (2.26), (2.37) and (2.38) to rewrite (3.31) - (3.33) for the case of the Primakoff solution:

$$(\nu-2)(\zeta-\xi_r) - S\Sigma = q\,\frac{\gamma+1}{\gamma-1}\,\frac{\bar{\rho}}{\rho}\,\zeta = q\zeta; \tag{3.34}$$

$$\dot{\xi}_r + \frac{2/t}{\nu(\gamma-1)+2}\,(\zeta-\xi_r) = \frac{2V_{1r}}{\gamma+1}\ ; \tag{3.35}$$

$$-\gamma p S\Sigma + \nu p(\zeta - \xi_r) = \frac{4\bar{\rho} S V V_{1r}}{\gamma+1} + qp\zeta . \tag{3.36}$$

By eliminating V_{1r} from (3.34) - (3.36) we can express ζ in erms of Σ,

$$\zeta = \frac{\gamma+1}{4(\nu-1)} S\Sigma, \tag{3.37}$$

and then substitute for ζ to obtain the desired boundary condition at R = S, expressed in terms of ξ alone:

$$(\gamma-1)\, S\Sigma + [\nu(\gamma-1) + 2]t\dot{\xi}_1 = 0. \tag{3.38}$$

Since by (2.16) $R\Sigma = r\sigma$, we can use (3.11) - (3.13) to write (3.38) in the form

$$(\gamma-1)b + [\nu(\gamma-1) + 2]\beta a = 0 . \tag{3.39}$$

Solving (3.14), (3.15) and (3.39) for a/b yields

$$-\frac{a}{b} = \frac{\frac{\gamma}{\nu}(\alpha+\nu-1) - 1}{\mu+\alpha}$$

$$= \frac{\mu-1 + [\frac{(\nu-2)\gamma}{\nu} - 1](\nu+\alpha-1) + \frac{\gamma}{\nu}[(\alpha+1)(\alpha+2)-\Lambda]}{(\alpha+1)(\alpha+2) - \Lambda}$$

$$= \frac{\gamma-1}{[\nu(\gamma-1)+2]\beta} . \tag{3.40}$$

Equations (3.40), together with (3.16), constitute a set of three algebraic relations in α, β and μ. The solutions are subject to an additional condition, namely that of "regularity" at the origin. This is the requirement that the second-order contribution to the total energy,

$$W_2 = \pi 2^{\nu-2} \int_0^S dRR^{\nu-1} \{ \rho \dot{\underline{\xi}}^2 + p[(\gamma-1)(\nabla_{\underline{R}} \cdot \underline{\xi})^2$$

$$+ \nabla_{\underline{R}} \underline{\xi} : \nabla_{\underline{R}} \underline{\xi}] \} \tag{3.41}$$

not diverge at R = 0. Using (2.16) to rewrite the integral in terms of r, we see that it is convergent provided

$$\mathrm{Re}\ \alpha > 1 - \nu, \tag{3.42}$$

which is also the condition that p_1 be finite at $r = 0$. Thus only modes satisfying (3.42) are physically realizable.

4. Solution of the eigenvalue problem

Equations (3.16) and (3.40) were solved for both $\nu = 2$ and $\nu = 3$. The degree of this system is not easy to determine by inspection, and initially solutions were obtained numerically for various choices of Λ. When it became apparent that there are only four solutions, the equations were manipulated to reduce them to a quartic in α, β or μ (these forms and their solutions were obtained largely by use of the interactive computerized symbolic manipulation system MACSYMA).

As our criterion of stability, we evaluated the time dependence of ζ/S. If ζ/S increases with time, the basic state is unstable; otherwise it is stable. From (2.28) and (3.11) we see that this ratio is proportional to

$$\frac{(S/f)^{\alpha}\, t^{\beta}}{f^{(\gamma+1)/2}} \sim f^{\frac{1}{2}[(\gamma-1)\alpha-\gamma-1]}t^{\beta}. \tag{4.1}$$

Hence the condition for stability is

$$\mathrm{Re}\,\Gamma \leq 0, \tag{4.2}$$

where

$$\Gamma = \frac{(\gamma-1)\ \alpha-\gamma-1}{\nu(\gamma-1)\ +\ 2} + \beta\ . \tag{4.3}$$

Accordingly we list below the values of Γ corresponding to those found for α and β.

(a) $\nu = 2$

The four roots of the α equation can be expressed in the form

$$(\gamma^2 - 1)\alpha = 1 \pm D \pm E^{\frac{1}{2}}\ , \tag{4.4}$$

where

$$D = [\gamma^2 - (\gamma^2-1)\Lambda]^{\frac{1}{2}} \tag{4.5}$$

and

$$E = \gamma^2(\gamma^2-1)(1-\Lambda) + 2(2-\gamma^2) \pm D. \tag{4.6}$$

Here the circled sign in (4.4) varies independently of the uncircled ones in (4.4) and (4.6), which are either both (+) or both (-). The corresponding values of β are given by

$$2\gamma(\gamma+1)\beta = \gamma(1 \pm D) \mp E^{\frac{1}{2}}, \tag{4.7}$$

while from (4.3) those of Γ are

$$\Gamma = -\frac{1}{2} \pm \frac{D}{2\gamma} \quad . \tag{4.8}$$

The associated values of μ, which we omit, can also be obtained as solutions of a quartic or by substitution in (3.16).

When $\gamma \to 1$ or when $\Lambda = 0$ or $\Lambda = 1$, D is real. The latter cases correspond to m = 0 and m = 1, the two flutelike modes with the longest wavelengths allowed in cylindrical geometry, and are therefore the ones expected (according to the simple picture discussed in Section 1) to be most unstable.

Table 2

$\gamma \approx 1$			
Root	$(\gamma-1)\alpha$	β	Γ
+ +	2	0	0
+ -	0	0	-1
- +	0	1	0
- -	0	0	-1

m = 0			
Root	α	β	Γ
+ +	1	$\frac{1}{\gamma}$	0
+ -	1	$\frac{1-\gamma}{\gamma}$	-1
- +	$\frac{3-\gamma}{\gamma-1}$	$\frac{\gamma-1}{\gamma}$	0
- -	$-\frac{3+\gamma}{\gamma+1}$	$\frac{\gamma-1}{\gamma(\gamma+1)}$	-1

m = 1			
Root	α	β	Γ
+ +	$\frac{2(1+\sqrt{2-\gamma^2})}{\gamma^2-1}$	$\frac{\gamma-\sqrt{2-\gamma^2}}{\gamma(\gamma+1)}$	$\frac{1-\gamma}{2\gamma}$
+ -	0	0	$-\frac{1+\gamma}{2\gamma}$
- +	$\frac{2(1-\sqrt{2-\gamma^2})}{\gamma^2-1}$	$\frac{\gamma+\sqrt{2-\gamma^2}}{\gamma(\gamma+1)}$	$\frac{1-\gamma}{2\gamma}$
- -	0	0	$-\frac{1+\gamma}{2\gamma}$

Table 2 lists the associated values of α, β and i for each of the four roots, labeled by the circled and uncircled signs. We note that (3.42) is satisfied for all solutions found in these limits except the (−,−) root when $m = 0$. Furthermore, $\Gamma \leq 0$ for all cases shown in the Table. For sufficiently large values of γ and Λ, D becomes imaginary and the real part of Γ always equals −1/2. Thus we have as a general conclusion $\Gamma \leq 0$, so that when $\nu = 2$ the blast waves solution under consideration are stable against perturbations with any value of m.

(b) $\nu = 3$

The analysis of the spherical case exactly parallels that carried out for $\nu = 2$. The four α roots are given by

$$2(\gamma^2-1)\alpha = 4-\gamma-\gamma^2 \pm D \pm E^{\frac{1}{2}}, \tag{4.9}$$

where now

$$D = [(3\gamma-1)^2 - 4(\gamma^2-1)\Lambda]^{\frac{1}{2}} \tag{4.10}$$

and

$$E = \gamma^4(9-4\Lambda) + 6\gamma^3 - \gamma^2(26-4\Lambda) - 18\gamma + 37$$
$$\pm 2(6-\gamma-3\gamma^2)D, \tag{4.11}$$

while the corresponding β roots are given by

$$2(\gamma+1)(3\gamma-1)\beta = 3\gamma-1 \pm \gamma D \mp E^{\frac{1}{2}}, \tag{4.12}$$

where the same conventions as before are employed with regard to circled and uncircled signs. Substitution of (4.9) and (4.12) in (4.3) yields

$$\Gamma = -\frac{1}{2} \pm \frac{D}{2(3\gamma-1)} \quad . \tag{4.13}$$

When $\gamma \to 1$, $D \to 2$. For $\ell = \Lambda = 0$, $D = 3\gamma - 1$, while for $\ell = 1$, $\Lambda = 2$ and $D = 3 - \gamma$. Table 3 shown the values of α, β and Γ in these limiting cases, where for conciseness we have written

$$F = \gamma^4 + 12\gamma^3 - 34\gamma^2 - 36\gamma + 73. \tag{4.14}$$

As before, (3.42) holds in all these cases except the $((-)-)$ mode for $\ell = 0$, and $\Gamma \leq 0$ for all cases shown in the Table. From (4.13), it is clear that the latter conclusion holds in the general case as well, although (3.42) is usually satisfied only for one pair of roots.

Table 3

$\gamma \approx 1$			
Root	$(\gamma-1)\alpha$	β	Γ
$\oplus +$	2	0	0
$\oplus -$	0	0	-1
$\ominus +$	0	1	0
$\ominus -$	0	0	-1

$\ell = 0$			
Root	α	β	Γ
$\oplus +$	1	$\frac{2}{3\gamma-1}$	0
$\oplus -$	1	$-\frac{3(\gamma-1)}{3\gamma-1}$	-1
$\ominus +$	$-\frac{2(\gamma-2)}{\gamma-1}$	$\frac{3(\gamma-1)}{3\gamma-1}$	0
$\ominus -$	$-2\,\frac{\gamma+3}{\gamma+1}$	$\frac{4(\gamma-1)}{(\gamma+1)(3\gamma-1)}$	-1

$\ell = 1$			
Root	α	β	Γ
$\oplus +$	$\frac{7-2\gamma-\gamma^2+F^{\frac{1}{2}}}{2(\gamma^2-1)}$	$\frac{-1+6\gamma-\gamma^2-F^{\frac{1}{2}}}{2(\gamma+1)(3\gamma-1)}$	$-\frac{2(\gamma-1)}{3\gamma-1}$
$\oplus -$	0	0	$-\frac{\gamma+1}{3\gamma-1}$
$\ominus +$	$\frac{7-2\gamma-\gamma^2-F^{\frac{1}{2}}}{2(\gamma^2-1)}$	$\frac{-1+6\gamma-\gamma^2+F^{\frac{1}{2}}}{2(\gamma-1)(3\gamma-1)}$	$-\frac{2(\gamma-1)}{3\gamma-1}$
$\ominus -$	-1	$\frac{\gamma-1}{3\gamma-1}$	$-\frac{\gamma+1}{3\gamma-1}$

5. Conclusion

The results presented in this paper establish rigorously the stability of a restricted class of the general Sedov solutions, This class(the Primakoff blast wave and its analogs) has been shown by Sedov (1959) to be degenerate. That is, the solution collapses to a single point in the phase plane defined by the reduced flow and sound speeds, as a consequence of the boundary condition being applied at a singularity(in this case, the origin). Oppenheim et al. (1972) make clear the relationship of such degenerate solutions to the general case.

The simplifying assumption responsible for the tractability of the problem we have solved is the requirement (2.36) that the unshocked medium have, for any given value of γ, a particular power-law density profile. As far as we know, there are no observations of supernovas for which the envelope gas density profiles are sufficiently accurately known to say whether they are close to those of the present model. It would be surprising if anything so simple actually occurred.

The results presented here provide no basis for any conclusions regarding the stability of the class of Sedov solutions as a whole. Nevertheless, it is probable that other profiles which are qualitatively similar, i.e., decreasing as some other power of the radius or perhaps logarithmically, are likewise stable. The reason for saying this is that (4.13) predicts positive(better than marginal) stability for almost all modes. Even a hypothetical shift in the direction of instability could well leave $\mathrm{Re}\Gamma < 0$ for all modes, or all except $\ell = 0$.

We note that the analysis of the cylindrical problem is similarly incomplete. It is incomplete in another way because of the omission of modes with $k_z \neq 0$. Although there are few astrophysical examples of cylindrical explosions, they do arise in laboratory experiments, such as those relating to the propagation of intense pulsed charged-particle beams through gas-filled chambers. There, too, the question of the stability of the resulting shock waves propagating into the surrounding medium has been widely discussed. In such experiments, however, the usual case of interest is that of radially increasing density profiles created as a result of channeling or hole-boring by previous pulses.

Note that our analysis applies, mutatis mutandis, to imploding shocks for which the flow behind (i.e., outside) the front is homologous. This is a restricted case of the general solution found by Guderley (1942). If we replace t by -t, all equations remain correct except those defining the energy, where the integrals must now be carried out from S to ∞. Instead of (2.44) we find a divergent expression for W. Equation (3.41) for W_2 can also diverge but must do so less rapidly. The condition for this, which is also the condition that first-order quantities be small compared with their unperturbed counterparts, is readily seen to be $\alpha \leq 1$, replacing (3.42). This means that a different subset of the solutions found in Section 4 corresponds to physically realizable modes. The general conclusion $\mathrm{Re}\Gamma \leq 0$ implies instability, since ζ/S diverges as $-t \to 0$. From Table 3 we see that the fastest growth ($\Gamma = -1$) is found as $\gamma \to 1$ or when $\ell = 0$, but we conclude from (4.13) that instability persists even when $\ell \to \infty$.

Considerably more work can probably be done along the lines of that presented here. While an analysis in closed form of the stability of the general Sedov solutions is unlikely, it would be surprising if the use of more powerful mathematical techniques than those employed here could not widen the range of accessible cases.

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References

1. Bernstein, I.B. and Book, D.L., 1978, Ap. J. 225, 635.
2. Bernstein, I.B. and Book, D.L., 1980, Ap. J. (in press)
3. Bernstein, I.B., Frieman, E.A., Kruskal, M.D., and Kulsrud, R.M., 1958, Proc. Roy. Soc. London Ser. A 244, 17.
4. Book, D.L., 1979, J. Fluid Mech. 95, 779.
5. Book. D.L. and Bernstein, I.B., 1979, Phys. Fluids 22, 79.
6. Chevalier, R.A., 1976, Ap. J. 207, 872.
7. Courant, R. and Friedrichs, K.O., 1948, Supersonic Flow and Shock Waves (Interscience Press, New York), p. 424.
8. D'yakov, S.P., 1954, ZhETF 27, 288.
9. Erpenbeck, J.J., 1962, Phys, Fluids 5, 1181.
10. Gardner, C.S. and Kruskal, M.D., 1964, Phys Fluids 7, 700.
11. Isenberg, P.A., 1977, Ap. J. 217, 597.
12. Keller, J.B., 1956, Q.J. Appl. Math. 14, 171.
13. Landau, L.D. and Lifshitz, E.M., 1959, Fluid Mechanics (Addison-Wesley, Reading, MA), p. 392.
14. Lerche, I. and Vasyliunas, V.M., 1976, Ap. J. 210, 85.
15. Newman, W.I., 1977, Astrophys. Space Sci. 47, 99.
16. Newman, W.I., 1979, Ap. J. (in press).
17. Oppenheim, A.K., Kuhl, A.L., Lundstrom, E. A., and Kamel, M.M. 1972, J. Fluid Mech. 52, 657.
18. Sedov.,L.I., 1959, Similarity and Dimensional Methods in Mechanics (Academic Press, New York), pp. 260-270.

19. Sedov, L. I., 1946, C. R. Acad. Sci. U.S.S.R 52, 17.

20. Taylor, G. I., 1950, Proc. Roy. Soc. London Ser. A 201, 159.

21. Von Neumann, J., 1947, Ch. 2 of Blast Wave, Los Alamos Sci. Lab. Tech. Series, Vol. VII, Pt. II, Bethe, H., Ed., LA-200; reprinted in Collected Works, Taub, A. H., Ed., Vol. VI (Pergamon Press, New York, 1963), p. 219.

22. Guderley, G., 1942, Luftfahrtforschung 19, 302.